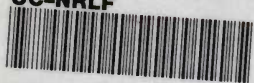


UC-NRLF



QB 564 845

LIBRARY
OF THE
UNIVERSITY OF CALIFORNIA.

Class

386
L2152

with it author's comments

Lamb

Class



QC665

L3

GENERAL

On the Reflection and Transmission of Electric Waves by a Metallic Grating.

BY PROF. HORACE LAMB, F.R.S.

[*Extracted from the Proceedings of the London Mathematical Society,*
Vol. XXIX., Nos. 644, 645.]



The main problem of this paper consists in the calculation of the disturbance produced in a train of electric waves by a plane grating composed of parallel, equal, and equidistant metallic strips. The treatment is approximate, and involves the assumption that the wave-length is large compared with the distance between the centres of consecutive strips; the application is, therefore, rather to Hertzian waves than to phenomena of ordinary Optics. The previous mathematical literature bearing directly on the subject consists of an investigation by Prof. J. J. Thomson,* and of two papers by Lord Rayleigh.† It will be seen that the author is under great obligations to each of these sources.

The work of §§ 1, 2, 3 is preliminary, and consists in the solution of several electrostatical problems. With modern methods these present no difficulty; but the results appeared to possess sufficient interest of their own to warrant the expenditure of a little trouble in the way of graphical representation. The diagrams obtained serve, moreover, to illustrate very clearly some of the conclusions arrived at afterwards in the discussion of the wave problem.

The problem of § 4 is introduced on account of its close connexion with what precedes, although it has no bearing on the main subject of the paper.

1. Let it be required to find the disturbance produced in a uniform electric field by a grating (of the kind above described) placed with its plane perpendicular to the lines of force. We will suppose that the axis of x is normal to the plane of the grating, and the axis of y in this plane, at right angles to the lengths of the strips.

In the case of a field symmetrical on the two sides of the grating, it will be found that (with a special choice of units) the potential- and stream-functions are given by the formula

$$\cosh w = \mu \cosh z, \quad (1)^\ddagger$$

* *Recent Researches on Electricity and Magnetism*, § 359. The problem attacked is that of reflection by a grating of cylindrical wires. Some notes on this question are included in the present paper, § 7.

† *Phil. Mag.*, April and July, 1897. See also *Theory of Sound*, § 292.

‡ [July, 1898.—This is not new, as the author had supposed. It was set in the *Mathematical Tripos*, Part II., 1895, and is due to Mr. Larmor.]

where $w = \phi + i\psi$, $z = x + iy$, (2)

and the constant μ is supposed greater than unity. This result was obtained originally by a modification of Schwarz's method, but it will be sufficient here to verify it *à posteriori*. The formula (1) gives

$$\left. \begin{aligned} \cosh \phi \cos \psi &= \mu \cosh x \cos y \\ \sinh \phi \sin \psi &= \mu \sinh x \sin y \end{aligned} \right\} \quad (3)$$

The locus $\phi = 0$ consists of those portions of the axis of y for which

$$1 > \mu \cos y > -1;$$

these represent the breadths of the metallic strips, so that (on the scale of our formulæ) the half-breadth of a strip is $\sin^{-1}(1/\mu)$. For other portions of the plane, ϕ is, so far, indeterminate as to sign; we will take it to be everywhere positive. Then ψ is indeterminate to the extent of a term $2s\pi$, but we may suppose that the lines $\psi = 0$, $\psi = \pm\pi$, $\psi = \pm 2\pi$, ... consist partly of the lines $y = 0$, $y = \pm\pi$, $y = \pm 2\pi$, ..., respectively, and partly of those portions of the axis of y for which

$$|\mu \cos y| > 1;$$

these correspond to the widths of the apertures between the successive strips. The half-breadth of an aperture, on the present scale, is $\cos^{-1}(1/\mu)$. At the centres of the apertures—for example, at the origin—we have $\phi = \cosh^{-1}\mu$; these are points of zero force, and the equipotential- and force-lines there meet at angles of 45° .

I have thought it worth while to trace the curves

$$\phi = \text{const.}, \quad \psi = \text{const.},$$

using the formula

$$\left. \begin{aligned} \cosh 2x \\ \cos 2y \end{aligned} \right\} = p \pm \sqrt{(p^2 - 2q + 1)}, \quad (4)$$

where

$$p = \frac{1}{2\mu^2} (\cosh 2\phi + \cos 2\psi), \quad q = \frac{1}{2\mu^2} (1 + \cosh 2\phi \cos 2\psi); \quad (5)$$

this is easily deduced from (3). The value of μ adopted for convenience of calculation was

$$\mu = \cosh \frac{1}{5}\pi = 1.2040,$$

whence $\sin^{-1}(1/\mu) = .312\pi$, $\cos^{-1}(1/\mu) = .188\pi$;

this gives the relative breadths of the strips and apertures. The curves are shewn in Fig. 1.

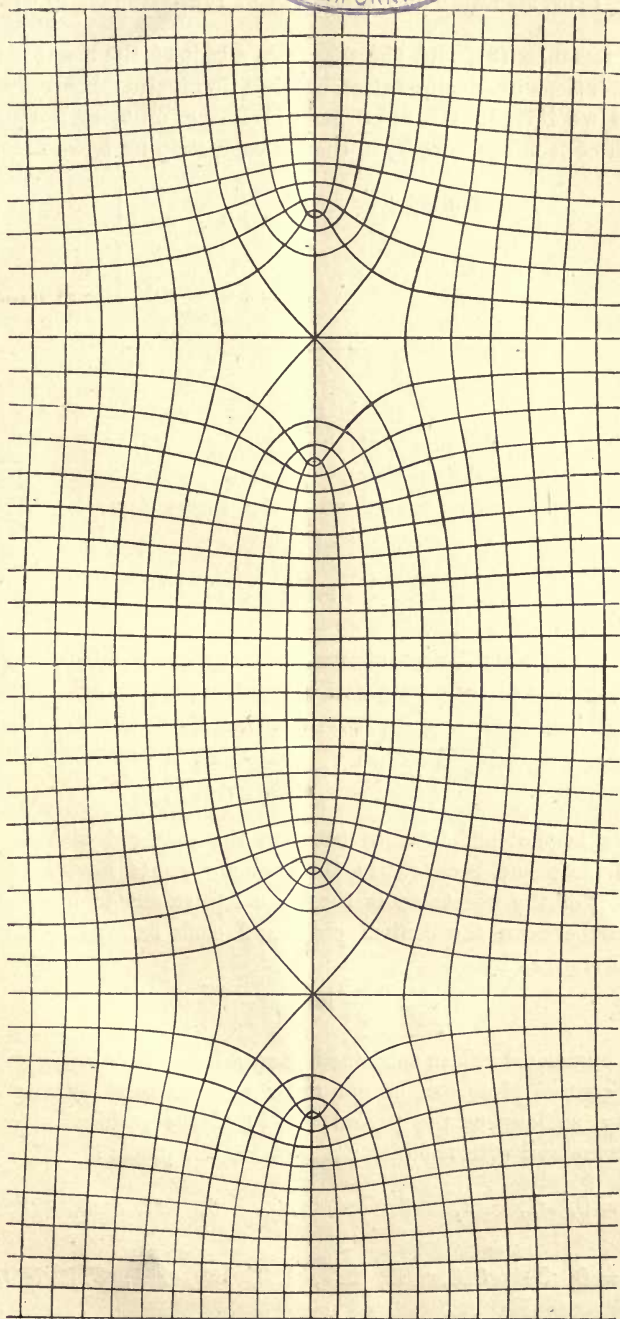


FIG. 1.

The formulæ (3), and the diagram, admit of the usual variety of interpretations in Electrostatics, Electric Conduction, Hydrodynamics, &c. If we introduce a more general linear unit, and denote the breadth of each aperture by a , that of each strip by b , we may write

$$\left. \begin{aligned} \cosh \phi \cos \psi &= \mu \cosh \frac{\pi x}{a+b} \cos \frac{\pi y}{a+b} \\ \sinh \phi \sin \psi &= \mu \sinh \frac{\pi x}{a+b} \sin \frac{\pi y}{a+b} \end{aligned} \right\}, \quad (6)$$

where
$$\mu = \sec \frac{\pi a}{2(a+b)} = \operatorname{cosec} \frac{\pi b}{2(a+b)}. \quad (7)$$

A first application of (6) is to find the capacity of a condenser formed by two metal plates at the same potential with an insulated grating half way between them, the distance between the two external plates being large compared with the interval $a+b$. For large positive values of x , the formulæ (6) give

$$\phi = \frac{\pi x}{a+b} + \log \mu, \quad \psi = \frac{\pi y}{a+b}. \quad (8)$$

The charge per unit area of one of the plates is $1/4(a+b)$ in the present units, and the capacity of the system per unit area is, therefore

$$\frac{1}{2\pi \left\{ x + \frac{a+b}{\pi} \log \sec \frac{\pi a}{2(a+b)} \right\}}, \quad (9)$$

where x is the half-distance between the external plates. If the central plate had been entire, the capacity would have been $1/2\pi x$; the effect of the apertures is therefore equivalent to increasing the distance between the central plate and each external plate by an amount c , given by

$$c = \frac{a+b}{\pi} \log \sec \frac{\pi a}{2(a+b)}. \quad (10)$$

The numerical values subjoined shew what a very large proportion of the central plate can be cut away without sensibly altering the capacity, so long as the period $(a+b)$ of the residual structure is large compared with the distance between the plates :

$$\frac{a}{a+b} = 0, \quad .1, \quad .2, \quad .3, \quad .4, \quad .5, \quad .6, \quad .7, \quad .8, \quad .9, \quad 1.0$$

$$\frac{c}{a+b} = 0, \quad .004, \quad .016, \quad .037, \quad .067, \quad .110, \quad .169, \quad .251, \quad .374, \quad .590, \quad \infty.$$

For another interpretation of (6) we may imagine that a current is flowing along a metal strip, of breadth $a+b$, which is interrupted by a central transverse non-conducting partition (e.g., a narrow cut in the metal), of breadth a , or by two lateral transverse cuts, opposite to one another, each of breadth $\frac{1}{2}a$. It appears that the additional resistance thus introduced is equal to that of a length $2c$ of the strip.

2. The function ϕ in (3) is a periodic even function of y , the period being π . It can therefore be expanded by Fourier's Theorem in a series of cosines of multiples of $2y$, the coefficients being functions of x whose general form is to be determined by substitution in the equation

$$\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} = 0. \quad (11)$$

Thus, for x positive we find, having regard to the conditions to be satisfied for large values of x ,

$$\phi = \log \mu + x + \sum_1^\infty C_m e^{-2mx} \cos 2my. \quad (12)$$

The precise values of the coefficients C_m hardly concern us, but it may be worth while to record them. The expansion of the function

$$w = \cosh^{-1}(\mu \cosh z) \quad (13)$$

can be effected directly, with the assistance of Lagrange's Theorem, and in this way we obtain the form (12), with

$$C_m = (-)^{m-1} \frac{1}{m} \left\{ 1 - \frac{m^2}{1^2 \cdot \mu^2} + \frac{m^2(m^2-1^2)}{1^2 \cdot 2^2 \cdot \mu^4} - \frac{m^2(m^2-1^2)(m^2-2^2)}{1^2 \cdot 2^2 \cdot 3^2 \cdot \mu^6} + \dots \right\}. \quad (14)$$

It may be noted that this contains $1-1/\mu^2$ as a factor.*

* [July, 1898.—We have

$$\begin{aligned} C_m &= (-)^{m-1} \frac{1}{m} F\left(m, -m, 1, \frac{1}{\mu^2}\right) \\ &= (-)^{m-1} \frac{1}{m} \left(1 - \frac{1}{\mu^2}\right) F\left(1+m, 1-m, 1, \frac{1}{\mu^2}\right), \end{aligned}$$

in the hypergeometric notation.

At the suggestion of one of the referees the proof of (14) is indicated. If we put $y = 0$ in (13), we have

$$\begin{aligned} \phi &= \cosh^{-1}(\mu \cosh x) = \log \{ \mu \cosh x + \sqrt{\mu^2 \cosh^2 x - 1} \} \\ &= \log \mu + x + \log \left[\frac{1+t}{2} + \sqrt{\left\{ \left(\frac{1+t}{2} \right)^2 - \frac{1}{\mu^2} \right\}} \right], \end{aligned}$$

where $t = e^{-2x}$. If we write

$$\zeta = \frac{1+t}{2} + \sqrt{\left\{ \left(\frac{1+t}{2} \right)^2 - \frac{1}{\mu^2} \right\}},$$

The comparison of (12) with (3) leads to a number of results in definite integrals. For example, using square brackets to denote that x has been put $= 0$ in the values of the enclosed functions, we have

$$\int_0^{\frac{1}{2}\pi} [\phi] dy = \frac{1}{2}\pi \log \mu, \quad (15)$$

$$\int_0^{\frac{1}{2}\pi} \left[\frac{d\phi}{dx} \right] dy = \frac{1}{2}\pi. \quad (16)$$

Now it appears from (3) that from $y = 0$ to $y = \cos^{-1}(1/\mu)$ we have

$$[\psi] = 0, \quad [\phi] = \cosh^{-1}(\mu \cos y),$$

whilst, from $y = \cos^{-1}(1/\mu)$ to $y = \frac{1}{2}\pi$,

$$[\phi] = 0, \quad [\psi] = \cos^{-1}(\mu \cos y).$$

Hence (15) gives the theorem

$$\int_0^{\beta} \cosh^{-1} \left(\frac{\cos y}{\cos \beta} \right) dy = \frac{1}{2}\pi \log \sec \beta. \quad (17)$$

This is easily verified. If we denote the value of the definite integral on the left hand by $f(\beta)$, we find

$$f'(\beta) = \tan \beta \int_0^{\beta} \frac{\cos y dy}{\sqrt{(\cos^2 y - \cos^2 \beta)}} = \frac{1}{2}\pi \tan \beta, \quad (18)$$

whence (17) follows by integration, since $f(0) = 0$.* The verification of (16) is still simpler; we have only to put $d\phi/dx = d\psi/dy$.

we find

$$\zeta = 1 + t \left(1 - \frac{1}{\mu^2 \zeta} \right) = 1 + t f(\zeta),$$

say. If x be positive, t will be less than unity, and $\log \zeta$ can be expanded in powers of t by Lagrange's Theorem, thus

$$\log \zeta = \log u + \sum_1^{\infty} \frac{t^m}{m!} \left(\frac{d}{du} \right)^{m-1} \frac{\{f(u)\}^m}{u},$$

where u is to be put $= 1$, finally. Working this out, we find

$$\log \zeta = \sum_1^{\infty} C_m t^m,$$

where C_m has the value given above. Hence the value of ϕ for $y = 0$ and $x > 0$ is

$$\phi = \log \mu + x + \sum_1^{\infty} C_m e^{-2mx}.$$

It easily follows that (12) gives the general value of ϕ for $x > 0$.]

* The theorem (17) is, substantially, one of a number given by Lobatschewsky, *Kasan Memoirs*, 1836 (Part I., p. 149), but the proof is quite different. The formula

$$\int_0^{\beta} \frac{y \sin y dy}{\sqrt{(\sin^2 \beta - \sin^2 y)}} = \frac{1}{2}\pi \log \sec \beta,$$

which is immediately deduced from (17) by a partial integration, was obtained by Legendre, *Exercices de Calcul Intégral*, t. i., Supplément, Case ix. (1811), as a particular case of a general theorem involving elliptic integrals.

3. When the field of force is not symmetrical with respect to the plane of the grating, the requisite formulæ are obtained by the addition of a term Az to w in (1), so that

$$\cosh (w + Az) = \mu \cosh z. \quad (19)$$

The most interesting special case is where the constant A is so chosen as to annul the force at a distance on one side, say that for which x is negative. We then have $A = -1$, and the equation (19) reduces to the form

$$\tanh z = \frac{\cosh w - \mu}{\sinh w}, \quad (20)$$

whence
$$z = \frac{1}{2} \log \frac{e^w - \mu}{\mu - e^{-w}}. \quad (21)$$

If we write $\mu = e^a$, this can be put into the shape

$$z = \frac{1}{2}w + \frac{1}{2} \log \frac{\sinh \frac{1}{2}(w-a)}{\sinh \frac{1}{2}(w+a)}, \quad (22)^*$$

or

$$x + iy = \frac{1}{2}(\phi + i\psi) + \frac{1}{2} \log \frac{\sinh \frac{1}{2}(\phi-a) \cos \frac{1}{2}\psi + i \cosh \frac{1}{2}(\phi-a) \sin \frac{1}{2}\psi}{\sinh \frac{1}{2}(\phi+a) \cos \frac{1}{2}\psi + i \cosh \frac{1}{2}(\phi+a) \sin \frac{1}{2}\psi}. \quad (23)$$

This gives, after a little manipulation,

$$\left. \begin{aligned} x &= \frac{1}{2}\phi + \frac{1}{4} \log \frac{\cosh(\phi-a) - \cos \psi}{\cosh(\phi+a) - \cos \psi}, \\ y &= \frac{1}{2}\psi + \frac{1}{2} \tan^{-1} \frac{\sinh a \sin \psi}{\cosh \phi - \cosh a \cos \psi} \end{aligned} \right\}. \quad (24)$$

By means of these equations the curves

$$\phi = \text{const.}, \quad \psi = \text{const.}$$

can be traced without much difficulty. Their forms are shown in Fig. 2 (p. 530) for the case of $a = \frac{1}{5}\pi$. The diagram affords the best verification of the formulæ, but the following points may be noticed.

In the first place, the line $\psi = 0$ consists of distinct portions, according as $\phi > a$. For $\phi > a$ we may take

$$y = 0, \quad x = \frac{1}{2}\phi + \frac{1}{2} \log \frac{\sinh \frac{1}{2}(\phi-a)}{\sinh \frac{1}{2}(\phi+a)}, \quad (25)$$

the value of x ranging from $-\infty$ to $+\infty$ as ϕ increases from a to ∞ . This gives the line of force through the origin, which is the centre of

* The solution for the present case was originally obtained in this form by Schwarz's method.

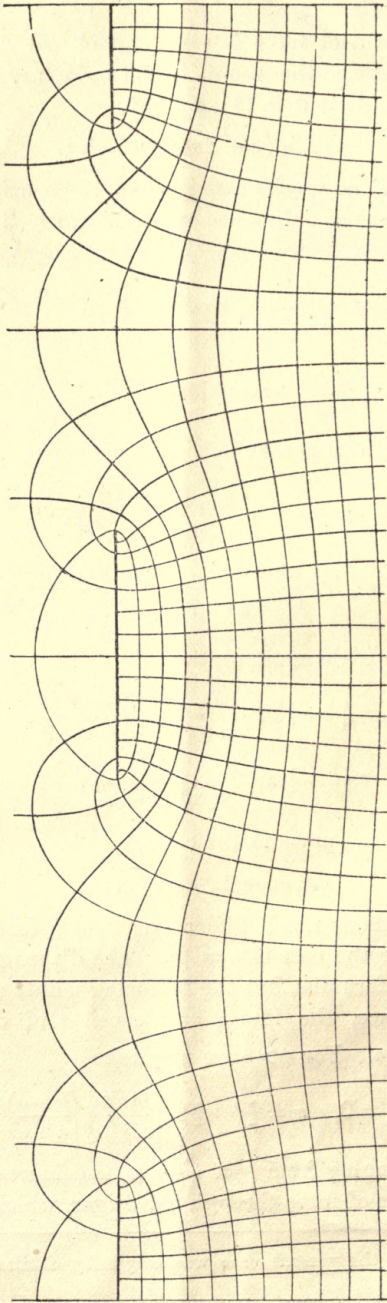


FIG. 2.

an aperture, and the lines of force through the centres of consecutive apertures are given in like manner by

$$\begin{aligned} y &= \pm\pi, \quad \pm 2\pi, \quad \pm 3\pi, \quad \dots, \\ \text{corresponding to} \quad \psi &= \pm 2\pi, \quad \pm 4\pi, \quad \pm 6\pi, \quad \dots \end{aligned}$$

If, on the other hand, $\psi = 0$ and $\phi < a$, we have

$$y = \frac{1}{2} \tan^{-1} 0, \quad x = \frac{1}{2} \phi + \frac{1}{2} \log \frac{\sinh \frac{1}{2} (a - \phi)}{\sinh \frac{1}{2} (a + \phi)}. \quad (26)$$

The value of x ranges from 0 to $-\infty$ as ϕ increases from 0 to a . The value to be attributed to $\tan^{-1} 0$ is determined by considerations of continuity. Thus, if we assign a very small fixed positive value to ψ , and then imagine ϕ to decrease from ∞ through a to 0, the fraction

$$\frac{\sinh a \sin \psi}{\cosh \phi - \cosh a \cos \psi}$$

will pass from a small positive value through infinity to a small negative value. Hence, if ψ approach zero from the positive side, the value of y in (26) will be $\frac{1}{2}\pi$. If ψ were to approach zero from the negative side, we should have $y = -\frac{1}{2}\pi$. The portions of the lines

$$y = \pm \frac{1}{2}\pi, \quad \pm \frac{3}{2}\pi, \quad \pm \frac{5}{2}\pi, \quad \dots$$

for which x is negative are lines of force abutting on the backs of the metal strips.

Again, for $\psi = \pi$, we have

$$y = \frac{1}{2}\pi, \quad x = \frac{1}{2} \phi + \frac{1}{2} \log \frac{\cosh \frac{1}{2} (\phi - a)}{\cosh \frac{1}{2} (\phi + a)}; \quad (27)$$

this value of x ranges from 0 to ∞ as ϕ increases from 0 to ∞ . And, generally, the portions of the lines

$$y = \pm \frac{1}{2}\pi, \quad \pm \frac{3}{2}\pi, \quad \pm \frac{5}{2}\pi, \quad \dots$$

for which x is positive are lines of force abutting on the fronts of the metal strips.

If in (24) we make $\phi = 0$, we get

$$x = 0, \quad y = \frac{1}{2}\psi + \frac{1}{2} \tan^{-1} \frac{\sinh a \sin \psi}{1 - \cosh a \cos \psi}. \quad (28)$$

This gives a series of isolated portions of the axis of y , representing the breadths of the metal strips. These are bounded by the points for which

$$\cos \psi = e^{-a} = 1/\mu,$$

and therefore

$$\frac{\sinh a \sin \psi}{1 - \cosh a \cos \psi} = \tan \psi. \quad (29)$$

For example, if we make ψ increase from 0 to π , y will diminish from $\frac{1}{2}\pi$ to the minimum value $\cos^{-1}e^{-a}$, and then increase again up to the value $\frac{1}{2}\pi$.

It is further to be noticed that the lines $\phi = \text{const.}$ fall into two systems separated by the curves $\phi = a$, which have the lines

$$y = 0, \pm\pi, \pm2\pi, \dots$$

as asymptotes. For $\phi < a$, the equipotential lines are closed. This is indicated clearly in the figure, which may be compared with the corresponding diagram for a grating of cylindrical wires given by Maxwell.*

The formulæ (24), when generalized as to the linear unit involved, lead to a determination of the capacity of a condenser formed of two parallel plates, one of which is entire, whilst the other is a grating. If the distance x between these plates be large compared with the interval $a + b$, where a and b have the same meanings as in § 1, we find for the capacity in question

$$\frac{1}{4\pi \left\{ x + \frac{a+b}{2\pi} \log \sec \frac{\pi a}{2(a+b)} \right\}}. \quad (30)$$

It is also easy to shew that the charge on the grating is divided between its front and back surfaces in the ratio $a + 2b : a$.

4. Although the matter is not very closely connected with the main object of this paper, it may be worth while to notice the interpretation of the formulæ (3) when the constant μ is *less* than unity. They give the distribution of electric force in a condenser one of whose plates has a series of parallel equidistant ribs projecting at right angles to its surface. Thus, if, altering the linear unit, we write

$$\cosh w = \mu \cosh \frac{\pi z}{a}, \quad (31)$$

$$\text{or} \quad \left. \begin{aligned} \cosh \phi \cos \psi &= \mu \cosh \frac{\pi x}{a} \cos \frac{\pi y}{a} \\ \sinh \phi \sin \psi &= \mu \sinh \frac{\pi x}{a} \sin \frac{\pi y}{a} \end{aligned} \right\}, \quad (32)$$

we find that the line $\phi = 0$ now consists of the whole of the axis of y , together with those portions of the lines

$$y = 0, \pm a, \pm 2a, \pm 3a, \dots$$

which extend from the axis of y to the points determined by

$$\cosh \frac{\pi x}{a} = \frac{1}{\mu}. \quad (33)$$

* *Electricity and Magnetism*, Vol. I., Fig. xiii.

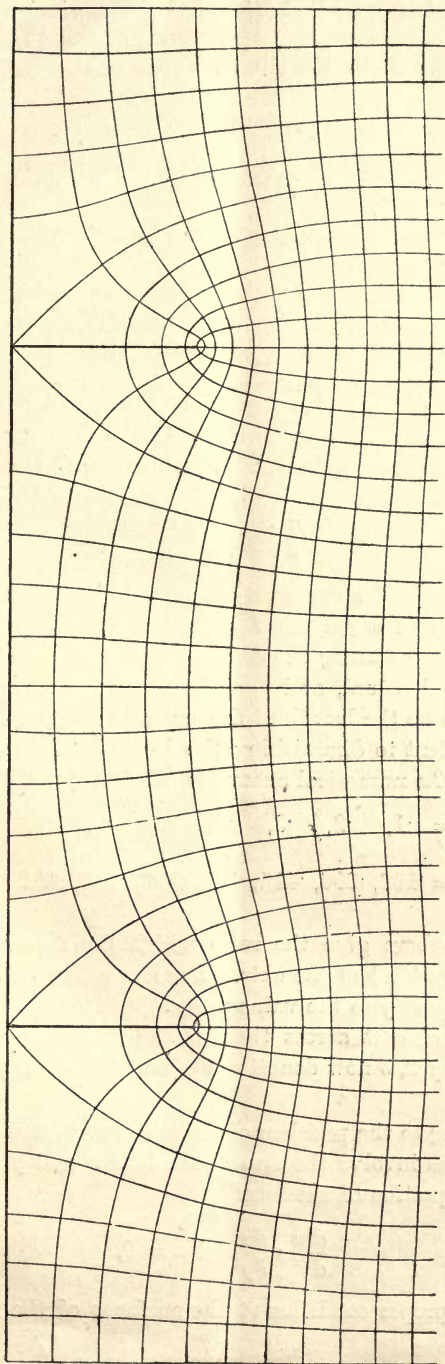


FIG. 3.

Hence, if b be the depth of a rib, we have

$$\mu = \operatorname{sech} \frac{\pi b}{a}. \quad (34)$$

The curves $\phi = \text{const.}$, $\psi = \text{const.}$ are shown in Fig. 3, for the case of $\mu = 1/\sqrt{2}$.

Proceeding as in § 1, we find that the capacity (per unit area) of a condenser of the above type is

$$\frac{1}{4\pi \left(x - \frac{a}{\pi} \log \cosh \frac{\pi b}{a} \right)}, \quad (35)$$

x denoting the distance between the plates, which is assumed to be large compared with b . The effect of the ribs in increasing the capacity is therefore equivalent to diminishing the distance between the plates by the amount

$$\frac{a}{\pi} \log \cosh \frac{\pi b}{a}. \quad (36)$$

Again, if we take the curves $\phi = \text{const.}$ to be lines of electric flow, we get the case of a current flowing along a relatively broad strip which is interrupted along one edge, or along the medial line, by a series of parallel, equal, and equidistant transverse cuts, one behind the other. If b be the length of the cuts, their effect (when at the edge) is equivalent to diminishing the breadth of the strips by the amount (36). As numerical examples, we find, for

$$b/a = \cdot 1, \cdot 2, \cdot 3, \cdot 4, \cdot 5, \cdot 6, \cdot 7, \cdot 8, \cdot 9, 1\cdot 0,$$

$$\frac{\log \cosh \pi b/a}{\pi b/a} = \cdot 156, \cdot 296, \cdot 415, \cdot 511, \cdot 586, \cdot 645, \cdot 691, \cdot 729, \cdot 756, \cdot 780.$$

The latter numbers give the ratio which the aforesaid diminution bears to the breadth b of the cuts. For increasing values of b/a this ratio tends obviously to the value unity.

In the case of cuts across the medial line the quantity (36) must be multiplied by 2, b now denoting the half-length of a cut.

5. Proceeding to the problem of electric waves, and assuming that all our functions involve the two dimensions x and y only, we have to satisfy an equation of the form

$$\frac{d^2 u}{dx^2} + \frac{d^2 u}{dy^2} + k^2 u = 0, \quad (37)$$

subject to the proper condition at the surfaces of the metal. If the

electrical vibrations be sufficiently rapid, the condition is that the component of electric force tangential to the surface, just outside the metal, must be zero.

We take separately the cases where the direction of the electric force is perpendicular and parallel, respectively, to the plane xy . In the former case, the equation (37) is satisfied by the electric force (R), and the condition which is to hold at the metal surfaces is $R = 0$, simply.

Let us now suppose that a train of waves of length $2\pi/k$ and amplitude unity is incident directly on the grating from the side on which x is positive. Omitting the time-factor e^{ikVt} , where V is the wave-velocity, this train may be represented symbolically by

$$R = e^{ikx}. \quad (38)$$

If in place of the grating we had an uninterrupted sheet of metal, the reflected wave would be represented by $-e^{-ikx}$, and the total disturbance on the side of x positive would be given by

$$R = e^{ikx} - e^{-ikx}, \quad (39)$$

for this satisfies (37) and makes $R = 0$ for $x = 0$.

In the actual circumstances we assume that, on the positive side of the grating,

$$R = e^{ikx} - e^{-ikx} + \chi, \quad (40)$$

and, on the negative side, $R = \chi'$, (41)

where χ and χ' must, of course, satisfy (37). Over the portions of the plane $x = 0$ which are occupied by the metal strips, we must have

$$\chi = 0, \quad \chi' = 0; \quad (42)$$

whilst the continuity of R and dR/dx demands that

$$\chi = \chi', \quad \frac{d\chi}{dx} + 2ik = \frac{d\chi'}{dx} \quad (43)$$

over the apertures. The latter condition will be satisfied provided we make

$$\frac{d\chi}{dx} = -ik, \quad \frac{d\chi'}{dx} = ik. \quad (44)$$

If the differential equation to be satisfied by χ had been

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} = 0, \quad (45)$$

in place of (37), the conditions relating to this function would (save as to the factor ik) have been the same as in the hydrodynamical problem where a system of equal, parallel, and equidistant

blades, all in the same plane, is moved perpendicular to itself in a frictionless liquid, which may have an independent general motion in the same direction. The function χ/ik would, in fact, give the value of the velocity-potential on the positive side of the plane $x = 0$. It must be borne in mind in this analogy that the blades correspond to the *apertures* of our grating, and the intervals between them to the metallic strips.* Now the *relative* motion of the fluid in such a case is shown in Fig. 1, to which the formulæ (6) apply; and a value of χ satisfying the conditions (42) and (44) would therefore be

$$\chi = -ikx + C\phi, \quad (46)$$

where ϕ is the positive function determined by (6).

This value of χ is necessarily periodic with respect to y , the period being $a+b$; and the line $y=0$ is a line of symmetry. Hence, expanding by Fourier's Theorem in a series of cosines of multiples of $2\pi y/(a+b)$, we find

$$\chi = C \log \mu + \left(\frac{\pi C}{a+b} - ik \right) x + C \sum_1^\infty C_m e^{-\kappa_m x} \cos \frac{2m\pi y}{a+b}, \quad (47)$$

where

$$\kappa_m = \frac{2m\pi}{a+b}, \quad (48)$$

and the first two terms have been determined from a comparison of (46) and (8). The terms involving $e^{\kappa_m x}$ are excluded by the condition of finiteness for $x = \infty$.

The true value of χ can be similarly expanded in a Fourier's series, the forms of the coefficients as functions of x being determined by means of the differential equation (37). The resulting form is

$$\chi = B_0 e^{-ikx} + \sum_1^\infty B_m e^{-\lambda_m x} \cos \frac{2m\pi y}{a+b}, \quad (49)$$

provided

$$\lambda_m^2 = \frac{4m^2\pi^2}{(a+b)^2} - k^2. \quad (50)$$

Since, by hypothesis, $a+b$ is small compared with the wave-length $2\pi/k$, the right-hand side of (50) is positive. Hence the quantities λ_m are real, and, moreover, differ respectively very little from κ_m . Terms involving $e^{\lambda_m x}$ are excluded as before by the condition of finiteness for $x = \infty$, so that the waves represented by χ are ultimately plane. The fact that they must travel outwards from the grating justifies the omission of the term in e^{ikx} .

* The foregoing method of analysing the conditions to be satisfied is taken from Lord Rayleigh's paper, *Phil. Mag.*, April, 1897.

For small values of x , the expansion (49) reduces to the form

$$\chi = B_0(1 - kx) + \sum_1^\infty B_m e^{-\kappa_m x} \cos \frac{2m\pi y}{a+b}, \quad (51)^*$$

approximately; and it appears, on comparison with (47), that the conditions incumbent on χ will all be satisfied, subject to an error of the order $k^2(a+b)^2/4\pi^2$, if

$$B_m = CC_m, \quad B_0 = C \log \mu = 1 + \frac{i\pi C}{k(a+b)}. \quad (52)$$

Substituting the value of μ from (7), we find

$$B_0 = \frac{ikc}{1+ikc}, \quad (53)$$

where c is the linear magnitude defined by (10).

The conditions for χ' are all fulfilled if we suppose that its value at any point P' on the negative side of the grating is equal, both in magnitude and in sign, to the value of χ at the *image* P of P' on the positive side. Hence the Fourier expansion of χ' is

$$\chi' = B_0 e^{ikx} + \sum_1 B_m e^{\lambda_m x} \cos \frac{2m\pi y}{a+b}. \quad (54)$$

In the above investigation, the coefficient of the primary wave has been taken to be unity. On the same scale, the coefficients of the reflected and transmitted waves are $-1+B_0$ and B_0 , or

$$-\frac{1}{1+ikc} \quad \text{and} \quad \frac{ikc}{1+ikc},$$

respectively. Hence the *intensities* I , I' (say) of these waves, in terms of that of the primary wave as unit, are given by

$$I = \frac{1}{1+k^2c^2}, \quad I' = \frac{k^2c^2}{1+k^2c^2}. \quad (55)$$

It appears from the numerical values given near the end of § 1 that the value of $c/(a+b)$ may be very moderate even when the breadth (b) of the metal strips is relatively small. Hence even a very open grating (*i.e.*, one in which a very small proportion of the area is occupied by metal) will produce almost total reflection of waves in which the electric displacement is parallel to the bars, provided the wave-length ($2\pi/k$) be moderately large compared with the period $a+b$ of the structure.

* Cf. J. J. Thomson, *loc. cit.*

The circumstances when the reflection is practically total are clearly illustrated in Fig. 2, where the lines $\phi = \text{const.}$ represent the lines of (alternating) magnetic force in the immediate neighbourhood of the grating. A similar interpretation applies also to Maxwell's diagram already referred to.

6. When the electric force is everywhere parallel to the plane xy , the magnetic force (γ) is everywhere perpendicular to this plane, and satisfies an equation of the form (37). The components of displacement-current in the dielectric are then

$$\frac{1}{4\pi} \frac{d\upsilon}{dy}, \quad -\frac{1}{4\pi} \frac{d\gamma}{dx}, \quad 0,$$

and the condition to be satisfied at the surfaces of the metal strips is $d\gamma/dn = 0$, where dn denotes an element of the normal.

Considering a train of waves represented by

$$\gamma = e^{ikx}, \quad (56)$$

and therefore incident directly on the grating from the side of x positive, we should have, if the apertures were filled up,

$$\gamma = e^{ikx} + e^{-ikx} \quad (57)$$

on the positive side, and $\gamma = 0$ on the negative side. In the actual case we assume, for the positive side,

$$\gamma = e^{ikx} + e^{-ikx} + \chi, \quad (58)$$

and, for the negative side, $\gamma = \chi'$, (59)

where χ and χ' must satisfy (37); and the conditions to be fulfilled over those portions of the line $x = 0$ which correspond to the metal strips are

$$\frac{d\chi}{dx} = 0, \quad \frac{d\chi'}{dx} = 0, \quad (60)$$

whilst over the apertures we must have

$$2 + \chi = \chi', \quad \frac{d\chi}{dx} = \frac{d\chi'}{dx}, \quad (61)$$

on account of the continuity of γ and $d\gamma/dx$. The former of the conditions (61) will be satisfied provided

$$\chi = -1, \quad \chi' = 1. \quad (62)$$

If the differential equation (37) were replaced by (45), the conditions applicable to χ would be similar to those which govern the

electric potential in the case of a grating placed transverse to the lines of force in a uniform field; but it must be remembered in this analogy that the bars in one problem correspond to the apertures in the other, and *vice versa*.* We should therefore have

$$\chi = -1 + C\phi, \quad (63)$$

where ϕ is determined by (6); but, if we still continue to denote by a and b the breadths of the apertures and of the strips, respectively, in our grating, the formula (7), giving the value of μ , must be replaced by

$$\mu = \sec \frac{\pi b}{2(a+b)} = \operatorname{cosec} \frac{\pi a}{2(a+b)}. \quad (64)$$

The Fourier expansion of (63) is

$$\chi = -1 + C \log \mu + \frac{\pi C}{a+b} x + C \sum_1^\infty C_m e^{-\kappa_m x} \cos \frac{2m\pi y}{a+b}, \quad (65)$$

whilst that of the true function χ has the form

$$\chi = C_0 e^{-ikx} + \sum_1^\infty B_m e^{-\lambda_m x} \cos \frac{2m\pi y}{a+b}, \quad (66)$$

where κ_m and λ_m are defined by (48) and (50). For small values of x the latter expansion takes the form

$$\chi = C_0 (1 - ikx) + \sum_1^\infty B_m e^{-\kappa_m x} \cos \frac{2m\pi y}{a+b}. \quad (67)$$

By comparison we find that (66) will satisfy approximately all the conditions of the problem, provided

$$B_m = CC_m, \quad C_0 = -1 + C \log \mu = \frac{i\pi C}{k(a+b)}. \quad (68)$$

Hence

$$C_0 = -\frac{1}{1 + ikc_1}, \quad (69)$$

provided

$$c_1 = \frac{a+b}{\pi} \log \sec \frac{\pi b}{2(a+b)}. \quad (70)$$

As regards χ' , all the conditions are satisfied if we suppose that its value at any point P' on the negative side of the grating is equal in absolute magnitude, but opposite in sign, to that of χ at the image P of P' on the positive side. Hence the Fourier expansion of χ' is

$$\chi' = -C_0 e^{ikx} - \sum_1^\infty B_m e^{\lambda_m x} \cos \frac{2m\pi y}{a+b}. \quad (71)$$

* See the footnote on p. 536.

The coefficients of the reflected and transmitted waves are $1 + C_0$ and $-C_0$, or

$$\frac{ikc_1}{1 + ikc_1} \quad \text{and} \quad \frac{1}{1 + ikc_1},$$

respectively, that of the primary being taken as unit. Hence the intensities J, J' of these waves are given by

$$J = \frac{k^2 c_1^2}{1 + k^2 c_1^2}, \quad J' = \frac{1}{1 + k^2 c_1^2}. \quad (72)$$

For sufficiently great wave-lengths there is very little reflection, even when the intervals a between the strips are very narrow compared with the breadths of the strips themselves.

* [These results, combined with those of § 5, serve to explain the polarizing efficacy of a grating, as employed originally by Hertz. As a numerical example, let us suppose that the wave-length is ten times the interval $a + b$. Then, even if one-tenth only of the area of the grating be occupied by the metal strips, the coefficients of reflection and transmission are, for electrical vibrations parallel to the lengths,

$$I = \cdot 879, \quad I' = \cdot 121.$$

The same numbers in reverse order give the coefficients of reflection and transmission for electrical vibrations transverse to the length, when the *apertures* occupy only one-tenth of the area; viz.,

$$J = \cdot 121, \quad J' = \cdot 879.$$

The coefficient of reflection in the one case, and of transmission in the other, increases rapidly with the wave-length.]

In the immediate neighbourhood of the grating the lines $\gamma = \text{const.}$ coincide with the curves $\phi = \text{const.}$ of Fig. 1. These give the directions of the electric oscillation; but it must be remembered that the portions of the vertical line of symmetry which represented the metal strips in the electrostatic problem of § 1 now correspond to apertures, and *vice versa*.

The mathematical conditions to which the magnetic force γ is subject in the problem of this section are exactly those which apply to the velocity-potential in the case of sound-waves incident directly on a grating of the kind considered.† It appears that a series of narrow slits, with relatively broad opaque intervals, may transmit a *very* large proportion of the original sound, provided the wave-length be sufficiently great.

* Paragraph inserted July, 1898.

† Cf. Rayleigh, *Phil. Mag.*, April, 1897, p. 272.

7. As mentioned in the introduction, the problem of reflection by a grating composed of parallel cylindrical bars has been discussed by Prof. J. J. Thomson in the case where the electric displacement is everywhere in the direction of the length. The result arrived at is that the reflection is total, without any limitation as to the thinness of the bars. It may be worth while to examine this point, and to complete the investigation by considering also the case where the electric displacement is transverse.

If, as in § 1, we write

$$w = \phi + i\psi, \quad z = x + iy, \quad (73)$$

the potential- and stream-functions corresponding to a system of equal and equidistant line-sources cutting the plane xy perpendicularly at the points $(0, 0)$, $(0, \pm a)$, $(0, \pm 2a)$, ..., are given by the formula

$$w \propto \log z + \log(z - ia) + \log(z + ia) + \log(z - 2ia) + \log(z + 2ia) + \dots, \quad (74)$$

or, say,

$$w = \log \sinh \frac{\pi z}{a}. \quad (75)$$

This makes

$$\left. \begin{aligned} \phi &= \frac{1}{2} \log \frac{1}{2} \left(\cosh \frac{2\pi x}{a} - \cos \frac{2\pi y}{a} \right) \\ \psi &= \tan^{-1} \left\{ \frac{\tan \frac{\pi y}{a}}{\tanh \frac{\pi x}{a}} \right\} \end{aligned} \right\}, \quad (76)$$

in agreement with Maxwell, *Electricity and Magnetism*, § 203.

The case of a row of *double* sources having their axes parallel to x is obtained by differentiating (75) with respect to z ; thus we find

$$w = \frac{\pi}{a} \log \coth \frac{\pi z}{a}, \quad (77)$$

$$\left. \begin{aligned} \phi &= \frac{\pi}{a} \frac{\sinh \frac{2\pi x}{a}}{\cosh \frac{2\pi x}{a} - \cos \frac{2\pi y}{a}} \\ \psi &= -\frac{\pi}{a} \frac{\sin \frac{2\pi y}{a}}{\cosh \frac{2\pi x}{a} - \cos \frac{2\pi y}{a}} \end{aligned} \right\}. \quad (78)$$

If the double sources have their axes parallel to y , we have merely to interchange the values of ϕ and ψ in these formulæ.

These results enable us to solve very simply a number of problems in Hydrodynamics, Magnetic Induction, and so on. For example, the stream-function for a liquid flowing through a grating of parallel cylindrical bars is

$$\psi = V \left\{ y - \frac{\pi b^2}{a} \frac{\sin \frac{2\pi y}{a}}{\cosh \frac{2\pi x}{a} - \cos \frac{2\pi y}{a}} \right\}, \quad (79)^*$$

where V is the velocity at infinity. This formula is easily verified, since, for small values of x and y , it reduces to the form

$$\psi = Vy \left(1 - \frac{b^2}{x^2 + y^2} \right). \quad (80)$$

It is here assumed that the radius b of a bar is small compared with the distance a between the axes of consecutive bars.

Again, in the case of a liquid flowing *parallel* to a row of cylindrical obstacles, we have

$$\psi = V \left\{ x - \frac{\pi b^2}{a} \frac{\sinh \frac{2\pi x}{a}}{\cosh \frac{2\pi x}{a} - \cos \frac{2\pi y}{a}} \right\}. \quad (81)$$

Returning to the formula (75), we find, if the real part of z be positive,

$$\begin{aligned} w &= \log \frac{1}{2} (e^{\pi z/a} - e^{-\pi z/a}) \\ &= \frac{\pi z}{a} - \log 2 - \sum_1^\infty \frac{1}{m} e^{-2m\pi z/a}, \end{aligned} \quad (82)$$

$$\text{whence} \quad \phi = \frac{\pi x}{a} - \log 2 - \sum_1^\infty \frac{1}{m} e^{-2m\pi x/a} \cos \frac{2m\pi y}{a}. \quad (83)$$

Since ϕ is an even function of x , we have, for x negative,

$$\phi = -\frac{\pi x}{a} - \log 2 - \sum_1^\infty \frac{1}{m} e^{2m\pi x/a} \cos \frac{2m\pi y}{a}. \quad (84)$$

For small values of x , we have, from (76),

$$\phi = \log \frac{\pi r}{a}, \quad (85)$$

where

$$r = \sqrt{(x^2 + y^2)}.$$

When the electric force (R) is everywhere parallel to the bars of the grating, we assume, for x positive,

* The curves $\psi = \text{const.}$ have been traced from this formula by Prof. Hele Shaw, *Trans. Inst. Nav. Arch.*, 1898.

$$R = e^{ikx} + Ae^{-ikx} + \sum_1^\infty D_m e^{-\lambda_m x} \cos \frac{2m\pi y}{a}, \quad (86)$$

and, for x negative,

$$R = Be^{ikx} + \sum_1^\infty D_m e^{\lambda_m x} \cos \frac{2m\pi y}{a}, \quad (87)$$

where λ_m is determined by

$$\lambda_m^2 = \frac{4m^2\pi^2}{a^2} - k^2. \quad (88)$$

For small values of x , we may ignore the difference between λ_m and $2m\pi/a$; and, if we further put $D_m = -C/m$, the formulæ (86) and (87) may be put into the shapes

$$R = 1 + A + ik(1-A)x + C\left(\phi - \frac{\pi x}{a} + \log 2\right), \quad (89)$$

$$\text{and} \quad R = B + ikBx + C\left(\phi + \frac{\pi x}{a} + \log 2\right), \quad (90)$$

respectively, where ϕ is defined by (83) and (84). The continuity of R and dR/dx requires that

$$1 + A = B, \quad ik(1-A) - \frac{\pi}{a}C = ikB + \frac{\pi}{a}C. \quad (91)$$

At the surface of the metal wires, where $r = b$ (say), we must have $R = 0$. This cannot be satisfied exactly; but, if the ratio b/a be sufficiently small, it is satisfied approximately if we make

$$B + C \log \frac{2\pi b}{a} = 0. \quad (92)^*$$

$$\text{If we now write} \quad c = \frac{a}{\pi} \log \frac{a}{2\pi b}, \quad (93)$$

we find, from (91) and (92), that

$$A = -\frac{1}{1+ikc}, \quad B = \frac{ikc}{1+ikc}. \quad (94)$$

The intensities of the reflected and transmitted waves, in terms of that of the primary wave, are therefore

$$= \frac{1}{1+k^2c^2}, \quad I' = \frac{k^2c^2}{1+k^2c^2}. \quad (95)$$

If the wave-length is at all large compared with c , kc is small, and the reflection is almost total. But, for any given wave-length (large compared with a), kc may be made as great as we please by sufficiently diminishing the radius b of the wires. In this way we can pass to the case of free transmission.

* [July, 1898.—Viz., this makes $R = 0$ for $x = 0$, $r = b$.]

When the electric force is everywhere parallel to the plane xy , it is convenient (as in § 6) to adopt as dependent variable the magnetic force (γ), which is everywhere perpendicular to this plane. We assume, for x positive,

$$\gamma = e^{ikx} + Ae^{-ikx} + \sum E_m e^{-\lambda_m x} \cos \frac{2m\pi y}{a}, \quad (96)$$

and, for x negative,

$$\gamma = Be^{ikx} - \sum E_m e^{\lambda_m x} \cos \frac{2m\pi y}{a}, \quad (97)$$

where λ_m is defined by (88). If we put $E_m = 2\pi C/a$, and ignore the difference between $\lambda_m x$ and $2m\pi x/a$, this gives, for small values of x ,

$$\gamma = 1 + A + ik(1-A)x + C \left(\frac{d\phi}{dx} - \frac{\pi}{a} \right), \quad (98)$$

and
$$\gamma = B + ikBx + C \left(\frac{d\phi}{dx} + \frac{\pi}{a} \right), \quad (99)$$

respectively. The continuity of γ and $d\gamma/dx$ on the two sides of the plane $x = 0$ requires that

$$1 + A - \frac{\pi}{a}C = B + \frac{\pi}{a}C, \quad 1 - A = B. \quad (100)$$

The surface-condition is that $d\gamma/dr = 0$ for $r = b$. If we put $x = r \cos \theta$, $d\phi/dx = \cos \theta/r$, this gives

$$ikB - \frac{C}{b^2} = 0. \quad (101)$$

Hence, if we write
$$c_1 = \frac{\pi b^2}{a}, \quad (102)$$

we get
$$A = \frac{ikc_1}{1 + ikc_1}, \quad B = \frac{1}{1 + ikc_1}. \quad (103)$$

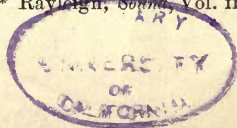
The intensities of the reflected and transmitted waves are now

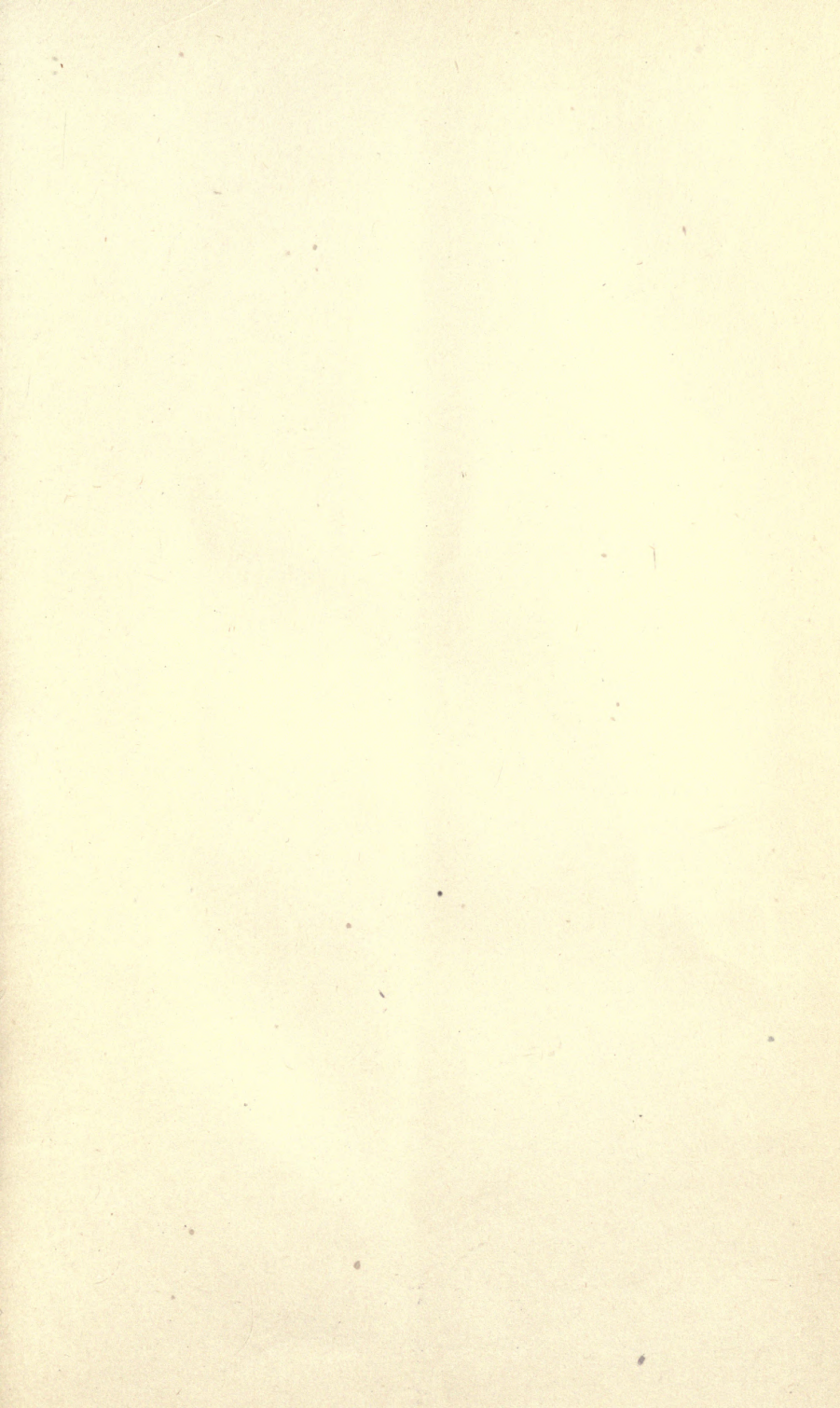
$$J = \frac{k^2 c_1^2}{1 + k^2 c_1^2}, \quad J' = \frac{1}{1 + k^2 c_1^2}. \quad (104)$$

If the half wave-length be large compared with b^2/a , we have free transmission, with hardly any reflection.

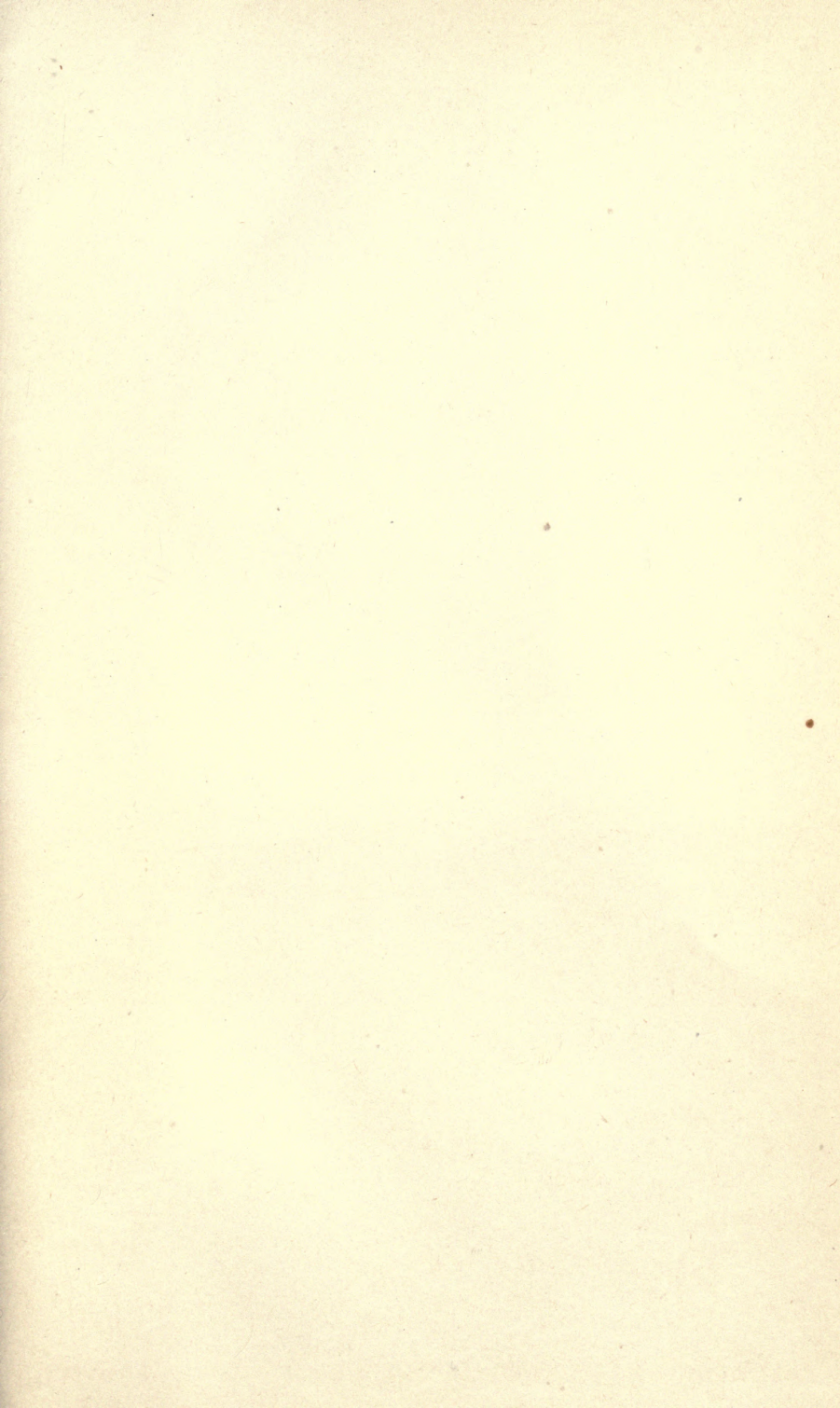
As in the problem of § 6, this last calculation has also an acoustical interpretation, and serves to further illustrate the "extreme smallness of the obstruction offered by fine wires or fibres to the passage of sound." *

* Rayleigh, *Sound*, Vol. II., § 343.









THIS BOOK IS DUE ON THE LAST DATE
STAMPED BELOW

AN INITIAL FINE OF 25 CENTS
WILL BE ASSESSED FOR FAILURE TO RETURN
THIS BOOK ON THE DATE DUE. THE PENALTY
WILL INCREASE TO 50 CENTS ON THE FOURTH
DAY AND TO \$1.00 ON THE SEVENTH DAY
OVERDUE.

SEP 6 1947

17 Nov '48

6 SEP '52 HD

SEP 8 1952 LU

20 Nov '52 MM

JUN 17 1953 LU

JAN 7 1966 94

REC'D LD

JAN 14 '66 -7 PM

132034

QC665

L3

